

# FINITE FIELD ELEMENTS OF HIGH ORDER ARISING FROM MODULAR CURVES (APPEARED IN *DESIGNS, CODES, AND CRYPTOGRAPHY*)

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**ABSTRACT.** In this paper, we recursively construct explicit elements of provably high order in finite fields. We do this using the recursive formulas developed by Elkies to describe explicit modular towers. In particular, we give two explicit constructions based on two examples of his formulas and demonstrate that the resulting elements have high order. Between the two constructions, we are able to generate high order elements in every characteristic. Despite the use of the modular recursions of Elkies, our methods are quite elementary and require no knowledge of modular curves. We compare our results to a recent result of Voloch. In order to do this, we state and prove a slightly more refined version of a special case of his result.

## 1. INTRODUCTION

Finding large order elements of finite fields has long been a problem of interest, particularly to cryptographers. Given a finite field  $\mathbb{F}_q$ , Gao [6] gives an algorithm for constructing elements of  $\mathbb{F}_{q^n}$  of order greater than

$$n^{\frac{\log_q n}{4 \log_q (2 \log_q n)} - \frac{1}{2}}.$$

The advantage of the algorithm is that it makes no restriction on  $q$  and it allows one to produce a provably high order element in any desired extension of  $\mathbb{F}_q$  provided that one can find a polynomial in  $\mathbb{F}_q[x]$  with certain desirable properties. Gao conjectures that for any  $n > 1$ , there exists a polynomial of degree at most  $2 \log_q n$  satisfying the conditions of his theorem. Conflitti has made some improvement to Gao's construction in [4]. However, the aforementioned conjecture remains unproven. Another result concerning the  $q$  "shifts" of an element of a general extension of  $\mathbb{F}_q$  appears in [12, Corollary 4.4].

For special finite fields, it is possible to construct elements which can be proved to have much higher orders. For example, in Theorems 1 and 2 of this paper we construct elements of higher order in extensions of  $\mathbb{F}_q$  of the form  $\mathbb{F}_{q^{2^n}}$  and  $\mathbb{F}_{q^{3^n}}$ . See [7, 8, 11] on orders of Gauss periods and [2, 3] on Kummer extensions. It has been pointed out to us that the method of [2, 3] is able to produce higher order elements in the same extensions as our method. However, our method of construction is new, and we hope that it will prove to be a fruitful technique.

In [14], Voloch shows that under certain conditions, one of the coordinates of a point on a plane curve must have high order. The bounds we obtain through our methods have order of magnitude similar to those predicted in the main theorem of [14]. In a special case however, Voloch is able to achieve bounds which are much better. See section 5 of [14]. Unfortunately, Voloch does not fully state this theorem and only alludes to how one may adapt the proof of his main theorem for this special case. The bounds given in [14] are not as explicit as the ones given in this paper. Moreover, Voloch gives no explicit examples of his theorems. In Section 6 of this paper, we apply Voloch's technique to obtain a more explicit version of the special case of his main theorem. We then construct a sequence of elements for which his bounds apply and compare with our methods.

In this paper, we consider elements in finite field towers recursively generated according to the equations for explicit modular towers [5]. We give two explicit constructions: one for odd characteristic and one for characteristic not equal to 3. In the first case, we explicitly construct elements of  $\mathbb{F}_{q^{2^n}}$  whose orders are bounded below by  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$ . In the second, we obtain elements of  $\mathbb{F}_{q^{3^n}}$  whose orders are bounded

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below by  $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$ . Throughout we use the convention that exponentiation is right-associative, i.e.,  $a^{b^c} := a^{(b^c)}$ .

## 2. CONSTRUCTIONS ARISING FROM MODULAR TOWERS

In [5], Elkies gives a recursive formula for the defining equations of the modular curve  $X_0(\ell^n)$  by identifying  $X_0(\ell^n)$  within the product  $(X_0(\ell^2))^{n-1}$  for  $n > 1$ . For several cases, he even writes explicit equations. For example, in the case  $\ell = 2$ , the recursion is governed by the rule

$$(x_j^2 - 1) \left( \left( \frac{x_{j+1} + 3}{x_{j+1} - 1} \right)^2 - 1 \right) = 1 \text{ for } j = 1, 2, \dots, n-2. \quad (1)$$

Elkies also notices that under a suitable change of variables and a reduction modulo 3, the equation becomes

$$y_{j+1}^2 = y_j - y_j^2,$$

which was used by Garcia and Stichtenoth [10] to recursively construct an asymptotically optimal function field tower. In fact, Elkies notes that many recursively constructed optimal towers may now be seen as arising from these modular curve constructions and speculates that perhaps all such towers are modular in this sense.

In this paper, we use Elkies' formulas to generate high order elements in towers of finite fields. For example, the following construction will yield high order elements in odd characteristic. The equation (1) may be manipulated to the form  $f(X, Y) = 0$ , where

$$f(X, Y) := Y^2 + (6 - 8X^2)Y + (9 - 8X^2), \quad (2)$$

and we have made the substitution  $X = x_j$  and  $Y = x_{j+1}$ . Now, choose  $q = p^m$  to be an odd prime power such that  $\mathbb{F}_q$  contains the fourth roots of unity (i.e.  $q \equiv 1 \pmod{4}$ ). Choose  $\alpha_0 \in \mathbb{F}_q$  such that  $\alpha_0^2 - 1$  is not a square in  $\mathbb{F}_q$ . In Lemma 3 (see Section 3), we will show that such an  $\alpha_0$  always exists. Finally, define  $\alpha_n$  by  $f(\alpha_{n-1}, \alpha_n) = 0$  for  $n \geq 1$ . This construction yields the following result; where, as usual, for a prime  $\ell$ ,  $\text{ord}_\ell(a)$  denotes the highest power of  $\ell$  dividing  $a$ .

**Theorem 1.** *Let  $\delta_n := \alpha_n^2 - 1$ . Then  $\delta_n$  has degree  $2^n$  over  $\mathbb{F}_q$ , and the order of  $\delta_n$  in  $\mathbb{F}_{q^{2^n}}$  is greater than  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1)}$  unless  $q \equiv 2 \pmod{3}$  and  $\alpha_0 = \pm(\frac{p-1}{2})$ , in which case the order of  $\delta_n$  is greater than  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1)-1}$ .*

To accommodate even characteristic, we have also considered Elkies' formula for  $X_0(3^n)$ . We will prefer to work with the equation in the polynomial form  $g(X, Y) = 0$ , where

$$g(X, Y) := Y^3 + (6 - 9X^3)Y^2 + (12 - 9X^3)Y + (8 - 9X^3). \quad (3)$$

For this construction, choose  $q$  to be a prime power congruent to 1 modulo 3 but not equal to 4. The condition  $q \equiv 1 \pmod{3}$  assures the presence of the third roots of unity in  $\mathbb{F}_q$ . Choose  $\beta_0 \in \mathbb{F}_q$  such that  $\beta_0^3 - 1$  is not a cube in  $\mathbb{F}_q$ . In Lemma 4 (see Section 3), we show that such a  $\beta_0$  always exists except when  $q = 4$ . Finally, define  $\beta_n$  by  $g(\beta_{n-1}, \beta_n) = 0$  for  $n \geq 1$ . For this construction, we have the following result.

**Theorem 2.** *Let  $\gamma_n := \beta_n^3 - 1$ . Then  $\gamma_n$  has degree  $3^n$  over  $\mathbb{F}_q$ , and the order of  $\gamma_n$  in  $\mathbb{F}_{q^{3^n}}$  is greater than  $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$ .*

There are two interesting things about the above constructions. The first is that, computationally, the elements  $\delta_n$  and  $\gamma_n$  appear to have much higher order than our bounds suggest. See Section 7 for examples. The second interesting thing is that, as with the case of the optimal function field tower constructions of Garcia and Stichtenoth [9, 10] arising from these modular curve recipes, our proofs do not at all exploit this modularity. Perhaps the key to achieving better bounds lies in this relationship.

The paper is organized as follows. In Section 3, we will state and prove some elementary number theory facts that will be of use to us. In Section 4, we consider the first construction; and in Section 5, we consider the second. Finally, in Section 7, we give a few examples of each of the main theorems.

### 3. NUMBER THEORETIC FACTS

Recall the following well known fact for detecting perfect  $n$ -th powers in finite fields. See [13, p. 81] for example.

**Fact 1.** *If  $q \equiv 1 \pmod{n}$ , then  $x \in \mathbb{F}_q^*$  is a perfect  $n$ -th power if and only if  $x^{(q-1)/n} = 1$ .*

Also recall the following facts, which can be easily proved.

**Fact 2.** *Let  $x \in \mathbb{F}_q^*$  of multiplicative order  $d$ . For  $m, n \in \mathbb{N}$ , if  $x^n \neq 1$  and  $x^{nm} = 1$ , then  $\gcd(d, m) > 1$ .*

**Fact 3.** *Let  $x \in \mathbb{F}_q^*$  of multiplicative order  $d$ . If  $\ell$  is a prime,  $m = \text{ord}_\ell(n)$ , and  $x^n$  is a nontrivial  $\ell$ -th root of unity, then  $\ell^{m+1}$  divides  $d$ .*

The following lemmas are useful for bounding the orders of the elements appearing in Theorems 1 and 2.

**Lemma 1.** *Let  $\ell, b \in \mathbb{N}$  such that  $b \equiv 1 \pmod{\ell}$ , and let  $M, N \in \mathbb{N}$  with  $M < N$ . Then*

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}, \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}\right) = \ell;$$

and hence  $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$  and  $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$  are coprime.

*Proof.* The following computation follows from Euclid's algorithm:

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}, b^{\ell^N} - 1\right) = \gcd\left(\ell, b^{\ell^N} - 1\right) = \ell. \quad (4)$$

Since  $M < N$ , repeatedly using the difference of  $\ell$ -th powers formula shows that  $\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$  divides  $b^{\ell^N} - 1$ . Also, since  $b \equiv 1 \pmod{\ell}$ , it is clear that  $\ell$  divides both  $\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}$  and  $\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$ . Therefore,

$$\gcd\left(\sum_{j=1}^{\ell} b^{\ell^M(\ell-j)}, \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}\right) = \ell.$$

□

**Lemma 2.** *Let  $\ell, b, N \in \mathbb{N}$  with  $\ell$  prime and  $b \equiv 1 \pmod{\ell}$ . If  $p$  is a prime dividing  $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$ , then  $p > \ell^{N+1}$ .*

*Proof.* Since  $\ell \geq 2$  and  $b \equiv 1 \pmod{\ell}$ ,  $\ell^2$  divides  $(b^{\ell^N} - 1)$ . Hence,  $p \neq \ell$  for otherwise, we have a contradiction with (4). Thus,  $p$  dividing  $\frac{1}{\ell} \sum_{j=1}^{\ell} b^{\ell^N(\ell-j)}$  implies that  $\sum_{j=1}^{\ell} b^{\ell^N(\ell-j)} \equiv 0 \pmod{p}$ . So,  $b^{\ell^N}$  is a nontrivial  $\ell$ -th root of unity modulo  $p$ . Therefore, by Fact 3,  $\ell^{N+1}$  divides  $p - 1$ , and hence  $p > \ell^{N+1}$ . □

The following two lemmas essentially give the necessary and sufficient conditions for completing the first step in the construction of our towers, i.e., under certain restrictions on  $q$ , they demonstrate the existence of  $\alpha_0$  and  $\beta_0$  each having its desired property. The proofs involve counting  $\mathbb{F}_q$  solutions to equations via character sums. We refer the reader to [13, Chapter 8] for more on this technique. As in [13], for characters  $\psi$  and  $\lambda$  on  $\mathbb{F}_q$ , we denote the Jacobi sum of  $\psi$  and  $\lambda$  by  $J(\psi, \lambda) := \sum_{a+b=1} \psi(a)\lambda(b)$ .

**Lemma 3.** *Let  $q$  be a prime power. Then there exists  $\alpha_0 \in \mathbb{F}_q$  such that  $\delta_0 = \alpha_0^2 - 1$  is not a square in  $\mathbb{F}_q$  if and only if  $q$  is odd.*

*Proof.* First, note that if  $q$  is even, then every element of  $\mathbb{F}_q$  is a square. So, we assume that  $q$  is odd. We desire  $\alpha_0 \in \mathbb{F}_q^*$  such that  $\alpha_0^2 - 1$  is not a square. Our method for proving that such an  $\alpha_0$  exists involves counting solutions to the equation  $x^2 - y^2 = 1$ . Let  $\tau$  be the unique character of exact order 2 on  $\mathbb{F}_q$ . Then

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^2 - y^2 = 1\} &= \sum_{\substack{a, b \in \mathbb{F}_q, \\ a+b=1}} \left( \sum_{j=0}^1 \tau^j(a) \right) \left( \sum_{j=0}^1 \tau^j(-b) \right) \\ &= \sum_{i=0}^1 \sum_{j=0}^1 \tau^j(-1) J(\tau^i, \tau^j) \\ &= q + \tau(-1) J(\tau, \tau) = q - 1. \end{aligned}$$

On the other hand, if  $\alpha_0^2 - 1$  is a square for all choices of  $\alpha_0$ , then  $\alpha_0^2 - 1 = y^2$  has a solution for all  $\alpha_0 \in \mathbb{F}_q$ . In this case, we have

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^2 - y^2 = 1\} &= \sum_{\alpha_0 \in \mathbb{F}_q} \#\{y \in \mathbb{F}_q : y^2 = \alpha_0^2 - 1\} \\ &= \sum_{\alpha_0^2=1} 1 + \sum_{\alpha_0^2 \neq 1} 2 = 2 + 2(q-2) = 2q-2. \end{aligned}$$

Thus, the assumption that  $\alpha_0^2 - 1$  is always a square leads to the conclusion  $q-1 = 2q-2$ , which implies  $q=1$ , a contradiction.  $\square$

**Lemma 4.** *Let  $q$  be a prime power. Then there exists  $\beta_0 \in \mathbb{F}_q$  such that  $\gamma_0 = \beta_0^3 - 1$  is not a cube in  $\mathbb{F}_q$  if and only if  $q \equiv 1 \pmod{3}$  and  $q \neq 4$ .*

*Proof.* First, note that if  $q \not\equiv 1 \pmod{3}$ , then every element of  $\mathbb{F}_q$  is a cube. So, we will assume that  $q \equiv 1 \pmod{3}$ . As mentioned earlier, this means that  $\mathbb{F}_q$  contains a primitive third root of unity. We now count  $\mathbb{F}_q$  solutions to the equation  $x^3 - y^3 = 1$ . Let  $\chi$  be any character of order 3 on  $\mathbb{F}_q$ .

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^3 - y^3 = 1\} &= \sum_{\substack{a, b \in \mathbb{F}_q, \\ a+b=1}} \left( \sum_{j=0}^2 \chi^j(a) \right) \left( \sum_{j=0}^2 \chi^j(-b) \right) \\ &= \sum_{i=0}^2 \sum_{j=0}^2 \chi^j(-1) J(\chi^i, \chi^j) \\ &= q - 2\chi(-1) + J(\chi, \chi) + J(\chi^2, \chi^2) \\ &= q - 2 + 2\operatorname{Re}J(\chi, \chi). \end{aligned}$$

On the other hand, if we assume that  $\beta_0^3 - 1$  is a cube for all choices of  $\beta_0 \in \mathbb{F}_q$ , then

$$\begin{aligned} \#\{(x, y) \in \mathbb{F}_q^2 : x^3 - y^3 = 1\} &= \sum_{\beta_0 \in \mathbb{F}_q} \#\{y \in \mathbb{F}_q : \beta_0^3 - y^3 = 1\} \\ &= \sum_{\beta_0^3=1} 1 + \sum_{\beta_0^3 \neq 1} 3 = 3 + 3(q-3) = 3q-6. \end{aligned}$$

Thus, the assumption that  $\beta_0^3 - 1$  is always a cube leads to the conclusion that  $|2q-4| = |(3q-6) - (q-2)| = |2\operatorname{Re}J(\chi, \chi)| \leq 2\sqrt{q}$ , which implies  $|q-2| \leq \sqrt{q}$ . This implies that  $(q-1)(q-4) \leq 0$ . The only  $q \equiv 1 \pmod{3}$  satisfying this inequality is  $q=4$ .  $\square$

#### 4. THE QUADRATIC TOWER FOR ODD CHARACTERISTIC

In this section, we consider the first tower, which is recursively constructed using (2). Throughout this section we will assume that  $p$  is an odd prime and that  $q = p^m \equiv 1 \pmod{4}$ . In particular, if  $p \equiv 3 \pmod{4}$ , then  $2|m$ . As discussed in the introduction, this condition ensures the existence of a primitive fourth root

of unity. This will be seen to be a necessary ingredient in the construction of our tower. We also fix  $\alpha_0$  such that  $\delta_0 = \alpha_0^2 - 1$  is not a square in  $\mathbb{F}_q$ . Recall that Lemma 3 ensures the existence of such an  $\alpha_0$ .

Before moving forward, we need to establish the relationship between  $\delta_n$  and  $\delta_{n-1}$ . From (2) and the definition of  $\delta_n$  (see Theorem 1), we deduce that  $\delta_{n-1}$  and  $\delta_n$  are related by  $F(\delta_{n-1}, \delta_n) = 0$  ( $n \geq 1$ ), where

$$F(X, Y) := Y^2 - (48X + 64X^2)Y - 64X. \quad (5)$$

We also fix the following more compact notation for the norm. We take

$$\begin{aligned} N_{n,j} : \mathbb{F}_{q^{2^n}} &\rightarrow \mathbb{F}_{q^{2^{n-j}}}, \\ \alpha &\mapsto \alpha^{\prod_{k=1}^j (q^{2^{n-k}} + 1)}. \end{aligned}$$

For the purpose of making the proof easier to digest, we break Theorem 1 into a pair of propositions.

**Proposition 1.** *The elements  $\alpha_n$  and  $\delta_n$  have degree 2 over  $\mathbb{F}_{q^{2^{n-1}}}$  for  $n \geq 1$ .*

*Proof.* First note that the discriminant of  $f(\alpha_{n-1}, Y)$  is  $\delta_{n-1} = \alpha_{n-1}^2 - 1$  for all  $n \geq 1$ . We will proceed by induction on  $n$ . Recall that  $\alpha_0$  was chosen so that  $\delta_0$ , the discriminant of  $f(\alpha_0, Y)$ , is not a square in  $\mathbb{F}_q$ . Thus,  $\alpha_1$  satisfies an irreducible polynomial of degree 2 over  $\mathbb{F}_q$ , i.e.,  $\alpha_1$  has degree 2 over  $\mathbb{F}_q$ . We may take  $\{1, \alpha_1\}$  as a basis for  $\mathbb{F}_q(\alpha_1)$  over  $\mathbb{F}_q$ . Writing  $\delta_1$  in terms of the basis, we have  $\delta_1 = \alpha_1^2 - 1 = (8\alpha_0^2 - 6)\alpha_1 + (8\alpha_0^2 - 10)$ . So,  $\delta_1 \in \mathbb{F}_q$  if and only if  $8\alpha_0^2 - 6 = 0$ . If  $8\alpha_0^2 - 6 = 0$ , then  $\delta_0 = \alpha_0^2 - 1 = -4^{-1}$ , which is a square in  $\mathbb{F}_q$  since  $\mathbb{F}_q$  contains the fourth roots of unity. This is contrary to our choice of  $\alpha_0$ . Thus,  $\delta_1$  has degree 2 over  $\mathbb{F}_q$  as well.

Now, suppose that  $\alpha_k$  and  $\delta_k$  both have degree 2 over  $\mathbb{F}_{q^{2^{k-1}}}$  for  $1 \leq k \leq n$ . Then  $f(\alpha_{n-1}, Y)$  is the minimum polynomial of  $\alpha_n$  over  $\mathbb{F}_{q^{2^{n-1}}}$ ; and hence, the discriminant is not a square in  $\mathbb{F}_{q^{2^{n-1}}}$ . In particular,

$$\delta_{n-1}^{(q^{2^{n-1}} - 1)/2} = -1. \quad (6)$$

Observe that  $F(\delta_{n-1}, Y)$  is the minimum polynomial of  $\delta_n$  over  $\mathbb{F}_{q^{2^{n-1}}}$ . To prove that the degree of  $\alpha_{n+1}$  over  $\mathbb{F}_{q^{2^n}}$  is 2, we show that  $f(\alpha_n, Y)$  is irreducible over  $\mathbb{F}_{q^{2^n}}$ . Now,

$$\begin{aligned} \delta_n^{(q^{2^n} - 1)/2} &= \left( \delta_n^{(q^{2^{n-1}} + 1)} \right)^{(q^{2^{n-1}} - 1)/2} = (N_{n,1}(\delta_n))^{(q^{2^{n-1}} - 1)/2} \\ &= (-64\delta_{n-1})^{(q^{2^{n-1}} - 1)/2} = -1. \end{aligned}$$

Here we have used (6) and the fact that  $-64$  is a square in  $\mathbb{F}_{q^{2^{n-1}}}$  since  $\mathbb{F}_q$  contains the fourth roots of unity. Thus,  $\delta_n$  is not a square, and hence  $f(\alpha_n, Y)$  is irreducible. So, the set  $\{1, \alpha_{n+1}\}$  forms a basis for  $\mathbb{F}_{q^{2^{n+1}}}$  over  $\mathbb{F}_{q^{2^n}}$ . Now, we write  $\delta_{n+1}$  in terms of the basis, and apply the same argument as for  $\delta_1$  to demonstrate that the degree of  $\delta_{n+1}$  over  $\mathbb{F}_{q^{2^n}}$  is 2 as well. This completes the induction and the proof.  $\square$

An easy induction proof, exploiting the fact that  $F(\delta_{k-1}, Y)$  is the minimum polynomial of  $\delta_k$  over  $\mathbb{F}_{q^{2^{k-1}}}$  for  $1 \leq k \leq n$ , shows that

$$N_{n,j}(\delta_n) = (-64)^{(2^j - 1)} \delta_{n-j} \quad (7)$$

for  $1 \leq j \leq n$ . This fact will be useful in the proof of the proposition below.

**Proposition 2.** *The order of  $\delta_n$  in  $\mathbb{F}_{q^{2^n}}$  is greater than  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1)}$  unless  $q \equiv 2 \pmod{3}$  and  $\alpha_0 = \pm(\frac{p-1}{2})$ , in which case the order of  $\delta_n$  is greater than  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$ .*

*Proof.* We first compute the power of 2 dividing the order of  $\delta_n$ . Recall from the proof of Proposition 1 that  $\delta_n^{(q^{2^n} - 1)/2} \neq 1$ ; but of course,  $\delta_n^{(q^{2^n} - 1)} = 1$  since  $\delta_n \in \mathbb{F}_{q^{2^n}}$ . Since  $q \equiv 1 \pmod{4}$ ,  $\text{ord}_2(q^{2^j} + 1) = 1$  for each  $j \geq 1$ . Repeatedly using the difference of squares formula, we have

$$\begin{aligned} \text{ord}_2\left(\frac{q^{2^n} - 1}{2}\right) &= \text{ord}_2(q - 1) - 1 + \sum_{j=0}^{n-1} \text{ord}_2(q^{2^j} + 1) \\ &= n - 1 + \text{ord}_2(q - 1). \end{aligned}$$

Thus,  $2^{n + \text{ord}_2(q-1)}$  divides the order of  $\delta_n$  by Fact 3.

Now we look for odd primes dividing the order. By Fact 2, the order of  $\delta_n$  has a common factor with  $(q^{2^{n-j}} + 1)/2$  for each  $j$  such that the  $\frac{(q^{2^n} - 1)}{(q^{2^{n-j}} + 1)/2}$  power of  $\delta_n$  is not equal to 1. By (7), we have that the  $\frac{(q^{2^n} - 1)}{(q^{2^{n-j}} + 1)/2}$  power of  $\delta_n$  is equal to

$$(\mathbb{N}_{n,j-1}(\delta_n))^{2(q^{2^{n-j}} - 1)} = ((-64)^{(2^{j-1}-1)}\delta_{n-j+1})^{2(q^{2^{n-j}} - 1)} = (\delta_{n-j+1})^{2(q^{2^{n-j}} - 1)} \neq 1$$

provided that  $\delta_{n-j+1}^2 \notin \mathbb{F}_{q^{2^{n-j}}}$ . From (5), we know that we may write  $\delta_{n-j+1}^2$  as

$$\delta_{n-j+1}^2 = (48\delta_{n-j} + 64\delta_{n-j}^2)\delta_{n-j+1} + 64\delta_{n-j}.$$

Thus,  $\delta_{n-j+1}^2 \in \mathbb{F}_{q^{n-j}}$  if and only if  $\delta_{n-j}$  satisfies the equation  $48\delta_{n-j} + 64\delta_{n-j}^2 = 0$ . If this were the case, then  $\delta_{n-j} = 0$  or  $\delta_{n-j} = -3^{-1}4$ . By Proposition 1, this implies that  $n = j$ . However,  $\delta_0 = 0$  contradicts the choice of  $\alpha_0$ ; and  $\delta_0 = -4^{-1}3$  contradicts the choice of  $\alpha_0$  unless  $-3$  is not a perfect square, that is, unless  $q \equiv 2 \pmod{3}$ . If  $q \equiv 2 \pmod{3}$ , then the only choices of  $\alpha_0$  that give  $\delta_0 = -4^{-1}3$  are  $\alpha_0 = \pm(\frac{p-1}{2})$ . Thus, the order of  $\delta_n$  has a common factor with  $(q^{2^{n-j}} + 1)/2$  for each  $1 \leq j \leq n$  unless  $q \equiv 2 \pmod{3}$ ,  $\alpha_0 = \pm(\frac{p-1}{2})$ , and  $j = n$ . Each of these factors must be odd since  $\text{ord}_2(q^{2^{n-j}} + 1) = 1$  as noted above. By Lemma 1 with  $\ell = 2$  and  $b = q$ , we see that these factors must be pairwise coprime as well. Hence, we get either  $n$  or  $n - 1$  distinct odd prime factors dividing the order of  $\delta_n$  depending on the case. By Lemma 2, each such prime factor must be bounded below by  $2^{n-j+1}$ . Therefore, the order of  $\delta_n$  is bounded below by

$$2^{n+\text{ord}_2(q-1)} \prod_{j=1}^n 2^{n-j+1} = 2^{n+\text{ord}_2(q-1)+n(n+1)/2} = 2^{\frac{n^2+3n}{2}+\text{ord}_2(q-1)}$$

unless  $q \equiv 2 \pmod{3}$  and  $\alpha_0 = \pm(\frac{p-1}{2})$ , in which case the order is bounded below by  $2^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_2(q-1) - 1}$ .  $\square$

Theorem 1 follows by combining the two propositions. The authors would like to point out that it is possible to achieve a slightly better lower bound for the order of  $\delta_n$  by the following method. First, choose a square root of  $\delta_{n-1}$ , say  $\sqrt{\delta_{n-1}} \in \mathbb{F}_{q^{2^n}}$ . Then use the method above to prove a lower bound for the order of  $\sqrt{\delta_{n-1}}$ . Finally, deduce a bound for the order of  $\delta_n$ . The improvement, however, only affects the coefficient of  $n$  in the exponent. Since computationally our bounds do not appear to be that close to the truth, we have decided to work directly with  $\delta_n$  instead.

## 5. THE CUBIC TOWER FOR CHARACTERISTIC NOT 3

In this section, we consider the second tower, which is recursively constructed using (3). Recall that, for this tower, we assume that  $q \equiv 1 \pmod{3}$  and  $q \neq 4$ . This means that  $\mathbb{F}_q$  will contain the third roots of unity, and hence the third roots of  $-1$  as well. We also fix a  $\beta_0$  such that  $\gamma_0 = \beta_0^3 - 1$  is not a cube in  $\mathbb{F}_q$ . Recall that Lemma 4 ensures the existence of such a  $\beta_0$ .

Before we begin the proof of Theorem 2, we need to establish the relationship between  $\gamma_{n-1}$  and  $\gamma_n$ . The relationship is given by  $G(\gamma_{n-1}, \gamma_n) = 0$  for  $n \geq 1$ , where

$$G(X, Y) := Y^3 - (270X + 972X^2 + 729X^3)Y^2 - (972X + 729X^2)Y - 729X. \quad (8)$$

This follows from (3) and the definition of  $\gamma_n$ . We also fix the following notation for the norm.

$$\begin{aligned} \mathbb{N}_{n,j} : \mathbb{F}_{q^{3^n}} &\rightarrow \mathbb{F}_{q^{3^{n-j}}}, \\ \beta &\mapsto \beta^{\prod_{k=1}^j \left( (q^{3^{n-k}})^2 + q^{3^{n-k}} + 1 \right)}. \end{aligned}$$

As in section 4, we break the result into two smaller propositions.

**Proposition 3.** *The elements  $\beta_n$  and  $\gamma_n$  both have degree 3 over  $\mathbb{F}_{q^{3^{n-1}}}$  for  $n \geq 1$ .*

*Proof.* By carefully examining the cubic formula applied to the polynomial, one observes that  $g(\beta_{n-1}, Y)$  is irreducible if and only if  $\gamma_{n-1} = \beta_{n-1}^3 - 1$  is not a cube in  $\mathbb{F}_{q^{3^{n-1}}}$ . Thus,  $\beta_n$  will have degree 3 over  $\mathbb{F}_{q^{3^{n-1}}}$  if and only if  $\gamma_{n-1}$  is not a cube in  $\mathbb{F}_{q^{3^{n-1}}}$  for all  $n \geq 1$ . As with the proof of Proposition 1, we proceed by

induction on  $n$ . Recall that  $\beta_0$  was chosen so that  $\gamma_0$  is not a cube in  $\mathbb{F}_q$ . Thus,  $\beta_1$  has degree 3 over  $\mathbb{F}_q$ . So, we may take  $\{1, \beta_1, \beta_1^2\}$  as a basis for  $\mathbb{F}_{q^3}$  over  $\mathbb{F}_q$ . Writing  $\gamma_1$  in terms of the basis, we have

$$\gamma_1 = \beta_1^3 - 1 = (9\beta_0^3 - 6)\beta_1^2 + (9\beta_0^3 - 12)\beta_1 + (9\beta_0^3 - 9).$$

So,  $\gamma_1 \in \mathbb{F}_q$  if and only if  $9\beta_0^3 - 6 = 0$  and  $9\beta_0^3 - 12 = 0$ . This leads to the conclusion that  $\gamma_0 = -3^{-1}$  and  $\gamma_0 = 3^{-1}$ , which implies that  $2 = 0$ , i.e., the characteristic is 2. In this case, we are led to the conclusion that  $\gamma_0 = 1$ , which is a cube. This of course is contrary to our choice of  $\gamma_0$ . Therefore,  $\gamma_1 \notin \mathbb{F}_q$ , i.e., the degree of  $\gamma_1$  over  $\mathbb{F}_q$  is 3. This completes the trivial case.

Now, let  $\omega$  be a primitive cube root of unity in  $\mathbb{F}_q$  and suppose that  $\beta_k$  and  $\gamma_k$  both have degree 3 over  $\mathbb{F}_{q^{3^{k-1}}}$  for  $1 \leq k \leq n$ . Then  $g(\beta_{n-1}, Y)$  is the minimum polynomial of  $\beta_n$  over  $\mathbb{F}_{q^{3^{n-1}}}$ ; and hence  $\gamma_{n-1}$  is not a cube in  $\mathbb{F}_{q^{3^{n-1}}}$ . In particular,

$$\gamma_{n-1}^{(q^{3^{n-1}}-1)/3} = \omega.$$

Observe that  $G(\gamma_{n-1}, Y)$  is the minimum polynomial of  $\gamma_n$  over  $\mathbb{F}_{q^{3^{n-1}}}$ . Thus,

$$\begin{aligned} \gamma_n^{(q^{3^n}-1)/3} &= \left( \gamma_n^{((q^{3^{n-1}})^2 + q^{3^{n-1}} + 1)} \right)^{(q^{3^{n-1}}-1)/3} = (N_{n,1}(\gamma_n))^{(q^{3^{n-1}}-1)/3} \\ &= (-729\gamma_{n-1})^{(q^{3^{n-1}}-1)/3} = \omega; \end{aligned}$$

i.e.,  $\beta_{n+1}$  has degree 3 over  $\mathbb{F}_{q^{3^n}}$ . To prove that  $\gamma_{n+1}$  also has degree 3 over  $\mathbb{F}_{q^{3^n}}$ , write  $\gamma_{n+1}$  in terms of the  $\mathbb{F}_{q^{3^n}}$ -basis  $\{1, \beta_{n+1}, \beta_{n+1}^2\}$ , and proceed as we did for  $\gamma_1$ .  $\square$

An easy induction proof using the fact that  $G(\gamma_{k-1}, Y)$  is the minimum polynomial of  $\gamma_k$  over  $\mathbb{F}_{q^{3^{k-1}}}$  for  $1 \leq k \leq n$ , shows that

$$N_{n,j}(\gamma_n) = (-729)^{(3^j-1)} \gamma_{n-j}$$

for  $1 \leq j \leq n$ .

**Proposition 4.** *The order of  $\gamma_n$  in  $\mathbb{F}_{q^{3^n}}$  is greater than  $3^{\frac{1}{2}n^2 + \frac{3}{2}n + \text{ord}_3(q-1)}$ .*

*Proof.* We first compute the power of 3 dividing the order of  $\gamma_n$ . Recall from the proof of Proposition 3 that  $\gamma_n^{(q^{3^n}-1)/3} \neq 1$ . However,  $\gamma_n^{(q^{3^n}-1)} = 1$  since  $\gamma_n \in \mathbb{F}_{q^{3^n}}$ . Since  $q \equiv 1 \pmod{3}$ ,  $\text{ord}_3((q^{3^j})^2 + q^{3^j} + 1) = 1$  for each  $j \geq 1$ . Repeatedly using the difference of cubes formula, we have

$$\begin{aligned} \text{ord}_3\left(\frac{q^{3^n}-1}{3}\right) &= \text{ord}_3(q-1) - 1 + \sum_{j=0}^{n-1} \text{ord}_3\left((q^{3^j})^2 + q^{3^j} + 1\right) \\ &= n - 1 + \text{ord}_3(q-1). \end{aligned}$$

Thus,  $3^{n+\text{ord}_3(q-1)}$  divides the order of  $\gamma$  by Fact 3.

Now, we look for primes dividing the order that are not equal to 3. In particular, we will show that the order of  $\gamma_n$  has a common factor with  $((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3$  for each  $1 \leq j \leq n$ . This factor must not be a multiple of 3 since  $\text{ord}_3((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1) = 1$  as noted above. By Lemma 1, with  $\ell = 3$  and  $b = q$ , we see that these factors must be pairwise coprime as well. Hence, we get  $n$  distinct prime factors dividing the order of  $\gamma_n$ , none of which are equal to 3. By Lemma 2, each of these primes must be bounded below by  $3^{n-j+1}$ . Hence, if we can show that the order of  $\gamma_n$  has a common factor with  $((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3$  for  $1 \leq j \leq n$ , then we have that the order of  $\gamma_n$  is bounded below by

$$3^{n+\text{ord}_3(q-1)} \prod_{j=1}^n 3^{n-j+1} = 3^{n+\text{ord}_3(q-1)+n(n+1)/2} = 3^{\frac{n^2+3n}{2}+\text{ord}_3(q-1)}.$$

By Fact 2, the proof will be complete when we show that the  $\frac{q^{3^n}-1}{((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3}$  power of  $\delta_n$  is not equal to 1 for  $1 \leq j \leq n$ . Now,  $\delta_n$  raised to the  $\frac{q^{3^n}-1}{((q^{3^{n-j}})^2 + q^{3^{n-j}} + 1)/3}$  power is equal to

$$(N_{n,j-1}(\gamma_n))^{3(q^{3^{n-j}}-1)} = ((-729)^{(3^{j-1}-1)} \gamma_{n-j+1})^{3(q^{3^{n-j}}-1)} \neq 1$$

provided  $\gamma_{n-j+1}^3 \notin \mathbb{F}_{q^{3^{n-j}}}$ . From (8), we know that we may write  $\gamma_{n-j+1}^3$  as

$$\gamma_{n-j+1}^3 = (270\gamma_{n-j} + 972\gamma_{n-j}^2 + 729\gamma_{n-j}^3)\gamma_{n-j+1}^2 + (972\gamma_{n-j} + 729\gamma_{n-j}^2)\gamma_{n-j+1} + 729\gamma_{n-j}.$$

Thus,  $\gamma_{n-j+1}^3 \in \mathbb{F}_{q^{3^{n-j}}}$  if only if  $\gamma_{n-j}$  satisfies the system

$$\begin{aligned} 270\gamma_{n-j} + 972\gamma_{n-j}^2 + 729\gamma_{n-j}^3 &= 0, \\ 972\gamma_{n-j} + 729\gamma_{n-j}^2 &= 0. \end{aligned}$$

Suppose that  $\gamma_{n-j}$  does satisfy the above system. If the characteristic is 2, the first equation implies that  $\gamma_{n-j} = 0$ , which is a contradiction. Suppose then that the characteristic is not 2. Solving the system, we have  $-3^{-2}(6 + \sqrt{6}) = \gamma_{n-j} = -3^{-1}4$ , where  $\sqrt{6}$  may be any square root of 6. This leads to the conclusion that  $30 = 0$ . Hence, the characteristic must be 5. By Proposition 3, we see that  $j = n$  since  $\gamma_{n-j} = -3^{-1}4 \in \mathbb{F}_q$ . However, this means that  $\gamma_0 = 2$ , which is in contradiction with the choice of  $\beta_0$  since 2 is a perfect cube in this case.  $\square$

## 6. COMPARISON WITH VOLOCH'S WORK

The following is an improvement of a result of Voloch [14, §5]. The proof is similar to the proof of the main theorem in [14], but more elementary in the sense that we avoid working with algebraic function fields.

**Theorem 3.** *Let  $q$  be a prime power, and let  $0 < \epsilon, \eta < 1$ . For  $d$  sufficiently large, if  $a \in \overline{\mathbb{F}}_q$  has order  $r$  and degree  $d$  over  $\mathbb{F}_q$  with  $r < d^{2-2\epsilon}$ , then  $a - 1$  has order at least  $\exp((1 - \eta)\frac{2\epsilon}{3}d^{\epsilon/3} \log d)$ . The degree  $d$  need only be large enough for the inequalities of (9) and (10) to hold, which depends only on the choices of  $\epsilon$  and  $\eta$ .*

*Proof.* Let  $0 < \epsilon < 1$  be given, and put  $N := \lceil d^{1-\epsilon} \rceil$ . Note that  $(r, q) = 1$  since  $r$  divides one less than a power of  $q$  and  $q$  is a prime power. Also, note that the elements  $a^{q^i}$ ,  $0 \leq i \leq d - 1$ , are distinct. It follows that the multiplicative order of  $q$  modulo  $r$  is exactly  $d$ . For each coset  $\Gamma$  of  $\langle q \rangle$  in  $(\mathbb{Z}/r\mathbb{Z})^*$ , we define  $J_\Gamma := \{n \leq N : n \bmod r \in \Gamma\}$ . Note that there are  $[(\mathbb{Z}/r\mathbb{Z})^* : \langle q \rangle] = \phi(r)/d$  cosets of  $\langle q \rangle$  in  $(\mathbb{Z}/r\mathbb{Z})^*$ . Now

$$\sum_{\Gamma} |J_\Gamma| = \#\{1 \leq n \leq N : \gcd(n, r) = 1\} = \frac{N\phi(r)}{r} + O(r^{\epsilon/10}),$$

where the sum is over all cosets of  $\Gamma$  in  $(\mathbb{Z}/r\mathbb{Z})^*$ . Thus, there exists a coset  $\Gamma = \gamma\langle q \rangle$  such that  $|J_\Gamma|$  is at least the average. That is,  $|J_\Gamma| \geq \frac{Nd}{r} + O(dr^{\epsilon/10}/\phi(r))$ . Thus, there exists a positive constant  $c_\epsilon$  so that  $|J_\Gamma| \geq \frac{Nd}{r} - c_\epsilon \frac{dr^{\epsilon/10}}{\phi(r)} \geq d^\epsilon - c_\epsilon d^{\frac{\epsilon-\epsilon^2}{5}}$  since  $d \leq \phi(r)$ .

Since  $\gamma$  is coprime to  $r$ , write  $\alpha\gamma + \beta r = 1$  and take  $c = a^\alpha$ . Then  $a = c^\gamma$ , and  $c$  has order  $r$  and degree at least  $d$ . Let  $b := a - 1$ . For each  $n \in J_\Gamma$ , there exists  $j_n$  such that  $n \equiv \gamma q^{j_n} \pmod{r}$ . Whence  $c^n = c^{\gamma q^{j_n}} = a^{q^{j_n}}$ , and so  $b^{q^{j_n}} = a^{q^{j_n}} - 1 = c^n - 1$ .

Now, for every  $I \subset J_\Gamma$  we write  $b_I := \prod_{n \in I} (c^n - 1) = \prod_{n_j \in I} b^{q^{n_j}}$  which is a power of  $b$ . Put  $T = \lceil d^{\epsilon/3} \rceil$ , and observe that for  $d$  sufficiently large

$$NT = \lceil d^{1-\epsilon} \rceil \lceil d^{\epsilon/3} \rceil < d. \quad (9)$$

We claim that for all distinct  $I, I' \subset J_\Gamma$  with  $|I| = |I'| = T$  we have that  $b_I \neq b_{I'}$ . Suppose that  $b_I = b_{I'}$ , and consider the non-zero polynomial

$$p(t) = \prod_{n \in I} (t^n - 1) - \prod_{n \in I'} (t^n - 1).$$

Observe that  $p(c) = b_I - b_{I'} = 0$ , and so  $\deg p(t) \geq \deg_{\mathbb{F}_q} c \geq d$ . On the other hand, we have that  $\deg p(t) \leq NT < d$ , a contradiction. Thus  $b_I \neq b_{I'}$  as claimed.

It follows that there are at least  $\binom{|J_\Gamma|}{T}$  distinct powers of  $b$ . Choose  $0 < \eta < 1$ . Then, for  $d$  sufficiently large,

$$\left( \frac{|J_\Gamma|}{T} \right) \geq \left( \frac{|J_\Gamma|}{d^{\epsilon/3}} - 1 \right)^{d^{\epsilon/3}} \geq \left( d^{2\epsilon/3} - c_\epsilon d^{-\frac{\epsilon(2+3\epsilon)}{15}} - 1 \right)^{d^{\epsilon/3}} \geq \exp \left( (1 - \eta) \frac{2\epsilon}{3} d^{\epsilon/3} \log d \right), \quad (10)$$

as required.  $\square$



In order to compare this result to Theorem 1, one may choose  $a = a_n$  to be a primitive  $2^n$ -th root of unity in  $\overline{\mathbb{F}}_q$ . The degree of  $a$  over  $\mathbb{F}_q$  will be  $2^{n - \text{ord}_2(q-1)}$ . Then, for  $n$  sufficiently large, the conditions of the above theorem will be satisfied. Similarly, one may choose  $a$  to be a primitive  $3^n$ -th root of unity in  $\overline{\mathbb{F}}_q$  to compare with Theorem 2.

Because of the requirement that  $a$  must have low order relative to its degree, there are many fields in which Theorem 3 will not apply. Furthermore, one may check that even though the bound of Theorem 3 will eventually dominate the bounds of Theorems 1 and 2, there will always be a range (in terms of  $n$ ) in which the bounds of Theorems 1 and 2 will be larger. For example, suppose we apply Theorem 3 to the case mentioned above, and we maximize the bound of Theorem 3 by setting  $\epsilon = 1$  and  $\eta = 0$ . Further, suppose we minimize the bound of Theorem 1 by say assuming that  $\text{ord}_2(q-1) = 1$ . Note that this will also serve to maximize the bound of Theorem 3. Under these assumptions, we may check that the bound of Theorem 1 will dominate for  $n \leq 11$ . However, we note that Theorem 3 does not actually apply if we choose  $\epsilon = 1$  and  $\eta = 0$ ; and the range of  $n$  for which Theorem 1 will dominate will be larger for any appropriate choice of  $\epsilon$  and  $\eta$ .

## 7. EXAMPLES OF THEOREMS

In this section we provide the data from the first several iterations for five examples of the main theorems: three for Theorem 1 and two for Theorem 2. The tables in this section provide information about the orders of  $\alpha_n$ ,  $\beta_n$ ,  $\delta_n$ , and  $\gamma_n$  in relation to our bound. We have chosen to take logs of these numbers because of their size. For each example, we note that the actual orders are much higher than our lower bounds. Computations were aided by MAGMA [1].

For our first example of Theorem 1, we choose  $q = 5$  and  $\alpha_0 = 2$ .

TABLE 1.  $q = 5$ ;  $\alpha_0 = 2$ .

$n$	$\log_2  \mathbb{F}_{5^{2n}}^* $	$\log_2  \langle \alpha_n \rangle $	$\log_2  \langle \delta_n \rangle $	$\log_2 \left( 2^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	4.59	4.59	3.00	3.00
2	9.28	9.28	7.70	6.00
3	18.6	16.0	17.0	10.0
4	37.1	35.6	31.5	15.0
5	74.2	69.8	68.6	21.0
6	148.	148.	143.	28.0
7	297.	295.	292.	36.0
8	594.	590.	589.	45.0

For our second example of Theorem 1, we choose  $q = 9$  and  $\alpha_0 = \zeta + 2$ , where  $\zeta$  is a root of  $x^2 + 1$ . Note that, in this example,  $\delta_n$  is actually primitive for each of the first eight iterations.

TABLE 2.  $q = 9; \alpha_0 = \zeta + 2$ .

$n$	$\log_2  \mathbb{F}_{9^{2^n}}^* $	$\log_2  \langle \alpha_n \rangle $	$\log_2  \langle \delta_n \rangle $	$\log_2 \left( 2^{\frac{1}{2}n^2 + \frac{3}{2}n + 3} \right)$
1	6.32	5.32	6.32	5.00
2	12.7	10.7	12.7	8.00
3	25.4	22.4	25.4	12.0
4	50.8	46.8	50.8	17.0
5	102.	96.5	102.	23.0
6	203.	197.	203.	30.0
7	406.	399.	406.	38.0
8	812.	804.	812.	47.0

For our final example of Theorem 1, we choose  $q = 121$  and  $\alpha_0 = \eta^8$ , where  $\eta$  is a root of  $x^2 + 7x + 2$ . Here,  $\delta_n$  is primitive except for  $n = 3$  and  $n = 7$ .

TABLE 3.  $q = 121; \alpha_0 = \eta^8$ .

$n$	$\log_2  \mathbb{F}_{121^{2^n}}^* $	$\log_2  \langle \alpha_n \rangle $	$\log_2  \langle \delta_n \rangle $	$\log_2 \left( 2^{\frac{1}{2}n^2 + \frac{3}{2}n + 3} \right)$
1	13.8	11.8	13.8	5.00
2	27.7	26.7	27.7	8.00
3	55.4	50.8	53.0	12.0
4	111.	109.	111.	17.0
5	222.	216.	222.	23.0
6	443.	440.	443.	30.0
7	886.	874.	883.	38.0

For our first example of Theorem 2, we choose  $q = 7$  and  $\beta_0 = 3$ . In this example,  $\gamma_n$  appears to alternate between being primitive and not.

TABLE 4.  $q = 7; \beta_0 = 3$ .

$n$	$\log_2  \mathbb{F}_{7^{3^n}}^* $	$\log_2  \langle \beta_n \rangle $	$\log_2  \langle \gamma_n \rangle $	$\log_2 \left( 3^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	8.42	7.41	5.84	4.76
2	25.3	25.3	25.3	9.52
3	75.8	75.8	74.2	15.8
4	228.	228.	228.	23.8
5	682.	681.	681.	33.3

For our second example of Theorem 2, we choose  $q = 16$  and  $\beta_0 = \xi$ , where  $\xi$  is a root of  $x^4 + x + 1$ . Note that here  $\gamma_n$  is primitive for each of the first five iterations.

TABLE 5.  $q = 16$ ;  $\beta_0 = \xi$ .

$n$	$\log_2  \mathbb{F}_{16^{3n}}^* $	$\log_2  \langle \beta_n \rangle $	$\log_2  \langle \gamma_n \rangle $	$\log_2 \left( 3^{\frac{1}{2}n^2 + \frac{3}{2}n + 1} \right)$
1	12.0	8.83	12.0	4.76
2	36.0	31.2	36.0	9.52
3	108.	102.	108.	15.8
4	324.	316.	324.	23.8
5	972.	962.	972.	33.3

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